On the solution of the static Maxwell system in axially symmetric inhomogeneous media

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Abstract

We consider the static Maxwell system with an axially symmetric dielectric permittivity and construct complete systems of its solutions which can be used for analytic and numerical solution of corresponding boundary value problems.

1 Introduction

Consider the static Maxwell system

$$\operatorname{div}(\varepsilon \mathbf{E}) = 0, \qquad \operatorname{rot} \mathbf{E} = 0 \tag{1}$$

where we suppose that ε is a function of the cylindrical radial variable $r = \sqrt{x_1^2 + x_2^2}$: $\varepsilon = \varepsilon(r)$. Two important situations are usually studied: the meridional field and the transverse field.

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The first case is characterized by the condition that the vector $\mathbf{E}$ is independent of the angular coordinate θ and the component E_θ of the vector $\mathbf{E}$ in cylindrical coordinates vanishes identically. The vector of such field belongs to a plane containing the axis x_3 and depends only on the distance r to this axis as well as on the coordinate x_3. The field then is completely described by a two-component vector-function in the plane (r, x_3).

The second case is characterized by the condition that the vector $\mathbf{E}$ is independent of x_3 and the component E_3 is identically zero. The vector of such field belongs to a plane perpendicular to the axis x_3 and the corresponding model reduces to a two-component vector-function in the plane (x_1, x_2).

In the present work in both cases we construct a complete system of solutions of the corresponding model. We use the fact that in both cases the system (1) reduces to a system describing so-called p-analytic functions [1], [4], [8], [10], [11], [13], [17], [18]

$$u_x = \frac{1}{p} v_y, \qquad u_y = -\frac{1}{p} v_x. \tag{2}$$

In the first case the function p is a function of one Cartesian variable x meanwhile in the second it is a function of $r = \sqrt{x^2 + y^2}$. In both cases we construct an infinite system of so-called formal powers [2], [5]. This is a complete system of exact solutions of equations (2) generalizing the system of usual complex powers $(z - z_0)^n$, $n = 0, 1, 2, \ldots$. Locally, near the center z_0 the formal powers behave asymptotically like powers. Nevertheless in general their behaviour can be arbitrarily different from that of powers but with a guarantee of their completeness in the sense that any solution of the considered equations can be represented as a uniformly convergent series of formal powers. The general theory of formal powers was developed by L. Bers [2] as a part of his pseudoanalytic function theory. Its application was restricted by the fact that only for a quite limited class of pseudoanalytic functions the explicit construction of formal powers was possible. L. Bers' results allow us to construct a complete system of formal powers in the meridional case. Nevertheless they are not applicable to the model arising from the transverse case. In the recent works [12] and [15] the class of solvable in this sense systems (2) was substantially extended. In the present work we use these results for solving the static Maxwell system in the transverse case. This combination of the relation between the static Maxwell system (1) and the system (2) together with the classical results of L. Bers on pseudoanalytic formal powers and new developments in [12] and [15] allow us to obtain a

general solution of the static Maxwell system in the axially symmetric case in the sense that we construct a complete system of its solutions for both the meridional and the transverse fields.

2 Reduction of the static Maxwell system to p-analytic functions

2.1 The meridional case

Introducing the cylindrical coordinates and making the assumptions that $\mathbf{E}$ is independent of the angular variable θ and that the component E_θ is identically zero we obtain that (1) can be written as follows

$$\frac{\partial E_r}{\partial x_3} - \frac{\partial E_3}{\partial r} = 0, \qquad \frac{1}{r}\frac{\partial(r\varepsilon E_r)}{\partial r} + \frac{\partial(\varepsilon E_3)}{\partial x_3} = 0.$$

Denote $x = r$, $y = x_3$, $u = E_3$ and $v = r\varepsilon E_r$. Then the system takes the form

$$u_x = \frac{1}{x\varepsilon(x)}v_y, \qquad u_y = -\frac{1}{x\varepsilon(x)}v_x,$$

where the subindices denote the derivatives with respect to the corresponding variables. Thus, in the case of a meridional field the vector $\mathbf{E}$ is completely described by an $x\varepsilon(x)$-analytic function $\omega = u + iv$.

2.2 The transverse case

We assume that $\mathbf{E}$ is independent of the longitudinal variable x_3 and $E_3 \equiv 0$. Then from (1) we have that the vector $(E_1, E_2)^T$ is the gradient of a function $u = u(x_1, x_2)$ which satisfies the two-dimensional equation

$$\operatorname{div}(\varepsilon \nabla u) = 0. \tag{3}$$

Denote $x = x_1$, $y = x_2$, $z = x + iy$ and consider the system

$$u_x = \frac{1}{\varepsilon}v_y, \qquad u_y = -\frac{1}{\varepsilon}v_x. \tag{4}$$

It is easy to see that if the function $\omega = u + iv$ is its solution then u is a solution of (3), and vice versa [14], if u is a solution of (3) in a simply connected domain Ω then choosing

$$v = \overline{A}(i\varepsilon u_{\overline{z}}), \tag{5}$$

where

$$\overline{A}\left[\Phi\right](x,y) = 2\left(\int_{\Gamma} \operatorname{Re}\Phi dx + \operatorname{Im}\Phi dy\right) + c, \tag{6}$$

c is an arbitrary real constant, Γ is an arbitrary rectifiable curve in Ω leading from (x_0, y_0) to (x, y) we obtain that $\omega = u + iv$ is a solution of (4). Here the subindex $\overline{z}$ means the application of the operator $\partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$. Note that due to the fact that u is a solution of (3) the function $\Phi = i\varepsilon u_{\overline{z}}$ satisfies the condition $\partial_y \operatorname{Re}\Phi - \partial_x \operatorname{Im}\Phi = 0$ and hence the integral in (6) is path-independent. For a convex domain then expression (6) can be written as follows

$$\overline{A}\left[\Phi\right](x,y) = 2\left(\int_{x_0}^{x} \operatorname{Re}\Phi(\eta, y)d\eta + \int_{y_0}^{y} \operatorname{Im}\Phi(x_0, \xi)d\xi\right) + c.$$

Note that v is a solution of the equation

$$\operatorname{div}(\frac{1}{\varepsilon}\nabla v) = 0.$$

Thus, equation (3) (and hence the system (1) in the case under consideration) is equivalent to the system (4) in the sense that if $\omega = u + iv$ is a solution of (4) then its real part u is a solution of (3) and vice versa, if u is a solution of (3) then $\omega = u + iv$, where v is constructed according to (5) is a solution of (4).

We reduced both considered cases the meridional and the transverse to the system describing p-analytic functions. In the first case $p = x\varepsilon(x)$ is a function of one Cartesian variable and in the second $p = \varepsilon(r)$, $r = \sqrt{x^2 + y^2}$. As we show below in both cases we are able to construct explicitly a complete system of formal powers and hence a complete system of exact solutions of the corresponding Maxwell system. Let us notice that equation (3) with ε being a function of the variable r was considered in the recent work [6] whith applications to electrical impedance tomography. The algorithm proposed in that work implies numerical solution of a number of ordinary differential equations arising after a standard separation of variables. Our construction of a complete system of solutions of (3) is based on essentially different ideas and does not require solving numerically any differential equation.

3 p-analytic functions and formal powers

3.1 The main Vekua equation

Consider the system describing p-analytic functions

$$u_x = \frac{1}{p} v_y, \qquad u_y = -\frac{1}{p} v_x, \tag{7}$$

where we suppose that p is a positive and continuously differentiable function of x and y. Together with tis system we consider the following Vekua equation which due to its importance in relation to second-order elliptic equations of mathematical physics is called [14], [15] the main Vekua equation

$$W_{\overline{z}} = \frac{f_{\overline{z}}}{f} \overline{W}, \tag{8}$$

where $f = \sqrt{p}$. The function $\omega = u + iv$ is a solution of (7) iff [15] $W = uf + iv/f$ is a solution of (8).

In [15] there was proposed a method for explicit construction of the system of formal powers corresponding to the main Vekua equation under a quite general condition on f. Here we briefly describe the method for which we need first to recall some basic definitions from L. Bers' theory of formal powers [2].

Let F and G be a couple of solutions of a Vekua equation

$$W_{\overline{z}} = a_{(F,G)} W + b_{(F,G)} \overline{W} \qquad \text{in } \Omega \tag{9}$$

such that $\operatorname{Im}(\overline{F}G) > 0$. Then (F, G) is said to be a *generating pair* corresponding to (9). The complex functions $a_{(F,G)}$ and $b_{(F,G)}$ are called *characteristic coefficients* of the pair (F, G) and it can be seen that

$$a_{(F,G)} = -\frac{\overline{F}G_{\overline{z}} - F_{\overline{z}}\overline{G}}{F\overline{G} - \overline{F}G}, \qquad b_{(F,G)} = \frac{FG_{\overline{z}} - F_{\overline{z}}G}{F\overline{G} - \overline{F}G}.$$

Together with these characteristic coefficients another pair of characteristic coefficients is introduced in relation to the notion of the (F, G)-derivative:

$$A_{(F,G)} = -\frac{\overline{F}G_z - F_z\overline{G}}{F\overline{G} - \overline{F}G}, \qquad B_{(F,G)} = \frac{FG_z - F_zG}{F\overline{G} - \overline{F}G},$$

where the z means the application of the operator $\partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$. As in the present work we do not use explicitly the notion of the (F,G)-derivative, we refer the interested reader to [2] for its definition and properties. However we do need the concept of characteristic coefficients for defining the following important object.

Definition 1 *Let (F,G) and (F_1,G_1) - be two generating pairs in Ω corresponding to the Vekua equations with coefficients $a_{(F,G)}$, $b_{(F,G)}$ and $a_{(F_1,G_1)}$ and $b_{(F_1,G_1)}$ respectively. Then (F_1,G_1) is called successor of (F,G) and (F,G) is called predecessor of (F_1,G_1) if*

$$a_{(F_1,G_1)} = a_{(F,G)} \qquad \textit{and} \qquad b_{(F_1,G_1)} = -B_{(F,G)}.$$

Definition 2 *A sequence of generating pairs $\{(F_m,G_m)\}$, $m = 0, \pm 1, \pm 2, \ldots$, is called a generating sequence if (F_{m+1},G_{m+1}) is a successor of (F_m,G_m). If $(F_0,G_0) = (F,G)$, we say that (F,G) is embedded in $\{(F_m,G_m)\}$.*

For any generating pair (F,G) the corresponding (F,G)-integral is defined as follows

$$\int_\Gamma w d_{(F,G)}\zeta = F(z)\operatorname{Re}\int_\Gamma G^* w d\zeta + G(z)\operatorname{Re}\int_\Gamma F^* w d\zeta$$

where Γ is a rectifiable curve leading from z_0 to z and (F^*,G^*) is an adjoint generating pair defined by the equations

$$F^* = -\frac{2\overline{F}}{F\overline{G} - \overline{F}G}, \qquad G^* = \frac{2\overline{G}}{F\overline{G} - \overline{F}G}.$$

If w is an (F_1,G_1) - pseudoanalytic function (i.e., it is a solution of the Vekua equation with the coefficients $a_{(F_1,G_1)}$ and $b_{(F_1,G_1)}$) then its (F,G)-integral is path-independent.

Now we are ready to introduce the definition of formal powers.

Definition 3 *The formal power $Z_m^{(0)}(a, z_0; z)$ with center at $z_0 \in \Omega$, coefficient a and exponent 0 is defined as the linear combination of the generators F_m, G_m with real constant coefficients λ, μ chosen so that $\lambda F_m(z_0) + \mu G_m(z_0) = a$. The formal powers with exponents $n = 1, 2, \ldots$ are defined by the recursion formula*

$$Z_m^{(n)}(a, z_0; z) = n\int_{z_0}^{z} Z_{m+1}^{(n-1)}(a, z_0; \zeta) d_{(F_m,G_m)}\zeta. \tag{10}$$

This definition implies the following properties.

1. $Z_m^{(n)}(a, z_0; z)$ is an (F_m, G_m)-pseudoanalytic function of z.

2. If a' and a'' are real constants, then $Z_m^{(n)}(a' + ia'', z_0; z) = a' Z_m^{(n)}(1, z_0; z) + a'' Z_m^{(n)}(i, z_0; z)$.

3. The asymptotic formulas
$$Z_m^{(n)}(a, z_0; z) \sim a(z - z_0)^n, \quad z \to z_0 \tag{11}$$
hold.

Writing $Z^{(n)}(a, z_0; z)$ we indicate that the formal power corresponds to the generating pair (F, G).

The definition of formal powers shows us that in order to obtain $Z^{(n)}(a, z_0; z)$ we need to have first the formal power $Z_1^{(n-1)}(a, z_0; z)$ for which it is necessary to calculate $Z_2^{(n-2)}(a, z_0; z)$ and so on. Thus, the problem of construction of formal powers of any order for a given generating pair (F, G) reduces to the construction of a corresponding generating sequence. Then definition 3 gives us a simple algorithm for constructing the formal powers. In other words, one needs a pair of exact solutions for each of the infinite number of Vekua equations corresponding to a generating sequence.

In the next subsection we show how this seemingly difficult task can be accomplished in a quite general situation. Meanwhile here we recall some well known results in order to explain that the system of formal powers in fact represents a complete system of solutions of a corresponding Vekua equation. First of all, let us notice that due to the property 2 of formal powers for every n (and for a fixed z_0) it is sufficient to construct only two formal powers: $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(i, z_0; z)$, then for any coefficient a the corresponding formal power $Z^{(n)}(a, z_0; z)$ is a linear combination of the former two.

An expression of the form $\sum_{n=0}^{N} Z^{(n)}(a_n, z_0; z)$ is called a *formal polynomial.* Under the conditions imposed in this work on the function ε and on the domain of interest Ω (see section 4 and for more details [15]) the following Runge-type theorem is valid where following [2] we say that a series converges normally in a domain Ω if it converges uniformly on every bounded closed subdomain of Ω.

Theorem 4 *[3] A pseudoanalytic function defined in a simply connected domain can be expanded into a normally convergent series of formal polynomials.*

In other words a pseudoanalytic function can be represented as an infinite linear combination of the functions

$$\left\{Z^{(n)}(1, z_0; z), \quad Z^{(n)}(i, z_0; z)\right\}_{n=0}^{\infty}.$$

Moreover, if we know that a pseudoanalytic function W satisfies the Hölder condition on the boundary of a domain of interest Ω (a common requirement when a boundary value problem is considered) then, e.g., the following estimate in the $C(\overline{\Omega})$-norm is available.

Theorem 5 *[16] Let W be a pseudoanalytic function in a domain Ω bounded by a Jordan curve and satisfy the Hölder condition on $\partial\Omega$ with the exponent α $(0 < \alpha \leq 1)$. Then for any $\epsilon > 0$ and any natural n there exists a pseudopolynomial of order n satisfying the inequality*

$$|W(z) - P_n(z)| \leq \frac{\text{Const}}{n^{\alpha-\epsilon}} \qquad \textit{for any } z \in \overline{\Omega}$$

where the constant does not depend on n, but only on ϵ.

These and other results on interpolation and on the degree of approximation by pseudopolynomials which can be found in the vast bibliography dedicated to pseudoanalytic function theory (see, e.g., [7], [9]) show us that the system of formal powers is as good for solving corresponding boundary value problems as is the system of usual complex powers $(z - z_0)^n$, $n = 0, 1, 2, \ldots$. The real (or imaginary) parts of $\{(z - z_0)^n\}_{n=0}^{\infty}$ are harmonic polynomials successfully applied to the numerical solution of boundary value problems for the Laplace equation. In a similar way the real parts of formal powers $\left\{Z^{(n)}(1, z_0; z), \quad Z^{(n)}(i, z_0; z)\right\}_{n=0}^{\infty}$ corresponding to the main Vekua equation (8) (where $f = \sqrt{\varepsilon}$) can be used for the numerical solution of boundary value problems for the conductivity equation (3) because as was shown in [14] the system of functions

$$\left\{\frac{1}{\sqrt{\varepsilon}} \operatorname{Re} Z^{(n)}(1, z_0; z), \quad \frac{1}{\sqrt{\varepsilon}} \operatorname{Re} Z^{(n)}(i, z_0; z)\right\}_{n=0}^{\infty}$$

is complete in the space of solutions of (3) in the sense of theorems 4 and 5.

A formal power $Z^{(n)}(a, z_0; z)$ related to the Vekua equation (8) corresponds to a formal power ${}_*Z^{(n)}(a, z_0; z)$ (we use the notation of L. Bers) related to the system (7) in the following way

$${}_*Z^{(n)}(a, z_0; z) = \frac{1}{f} \operatorname{Re} Z^{(n)}(a, z_0; z) + if \operatorname{Im} Z^{(n)}(a, z_0; z).$$

As we will see in the meridional case it is convenient to work directly with formal powers ${}_{*}Z^{(n)}(a, z_0; z)$. Any solution $\omega = u + iv$ of the system (7) can be expanded into a normally convergent series of real linear combinations of the complex functions $\left\{{}_{*}Z^{(n)}(1, z_0; z), \quad {}_{*}Z^{(n)}(i, z_0; z)\right\}$.

3.2 Construction of generating sequences

In [15] the following result was obtained.

Theorem 6 *Let $F = U(u)V(v)$ and $G = \frac{i}{U(u)V(v)}$ where U and V are arbitrary differentiable nonvanishing real valued functions, $\Phi = u + iv$ is an analytic function of the variable $z = x + iy$ in Ω such that Φ_z is bounded and has no zeros in Ω. Then the generating pair (F, G) is embedded in the generating sequence (F_m, G_m), $m = 0, \pm 1, \pm 2, \ldots$ in Ω defined as follows*

$$F_m = (\Phi_z)^m F \quad \text{and} \quad G_m = (\Phi_z)^m G \quad \text{for even } m$$

and

$$F_m = \frac{(\Phi_z)^m}{U^2} F \quad \text{and} \quad G_m = (\Phi_z)^m U^2 G \quad \text{for odd } m.$$

This theorem opens the way for construction of generating sequences and consequently of formal powers in a quite general situation (see [15]) and in particular in both cases considered in the present work. In the meridional case the theorem reduces to the result of L. Bers [2] which we use in the next subsection while in the transverse case this and other classical results are insufficient for constructing formal powers explicitly and theorem 6 is indispensable.

4 Construction of formal powers

4.1 Formal powers in the meridional case

As was shown in subsection 2.1 in the meridional case the Maxwell system reduces to the following couple of equations

$$u_x = \frac{1}{x\varepsilon(x)} v_y, \qquad u_y = -\frac{1}{x\varepsilon(x)} v_x$$

which is equivalent to the system considered in [2, N18.1]

$$\sigma(x)\phi_x = \tau(y)\psi_y, \qquad \sigma(x)\phi_y = -\tau(y)\psi_x.$$

Taking $\sigma(x) = x\varepsilon(x)$ and $\tau \equiv 1$ we can use the elegant formulas for the generating powers obtained by L. Bers. Let

$$X^{(0)}(x_0, x) = \widetilde{X}^{(0)}(x_0, x) = 1$$

and for $n = 1, 2, ...$denote

$$X^{(n)}(x_0, x) = n\int_{x_0}^{x} X^{(n-1)}(x_0, t)\frac{1}{t\varepsilon(t)}dt \qquad \text{for odd } n$$

$$X^{(n)}(x_0, x) = n\int_{x_0}^{x} X^{(n-1)}(x_0, t)t\varepsilon(t)dt \qquad \text{for even } n$$

$$\widetilde{X}^{(n)}(x_0, x) = n\int_{x_0}^{x} \widetilde{X}^{(n-1)}(x_0, t)t\varepsilon(t)dt \qquad \text{for odd } n$$

$$\widetilde{X}^{(n)}(x_0, x) = n\int_{x_0}^{x} \widetilde{X}^{(n-1)}(x_0, t)\frac{1}{t\varepsilon(t)}dt \qquad \text{for even } n$$

Then the formal powers in the meridional case are given by the expressions

[2]

$$\begin{aligned} {}_*Z^{(n)}(a' + ia'', z_0; z) = a' &\sum_{k=0}^{n} \binom{n}{k} X^{(n-k)} i^k (y - y_0)^k \\ &+ ia'' \sum_{k=0}^{n} \binom{n}{k} \widetilde{X}^{(n-k)} i^k (y - y_0)^k \qquad \text{for odd } n \end{aligned}$$

and

$$
{}_*Z^{(n)}(a'+ia'',z_0;z) = a'\sum_{k=0}^{n}\binom{n}{k}\widetilde{X}^{(n-k)}i^k(y-y_0)^k + ia''\sum_{k=0}^{n}\binom{n}{k}X^{(n-k)}i^k(y-y_0)^k \qquad \text{for even } n.
$$

4.2 Formal powers in the transverse case

As was shown in subsection 2.2 the Maxwell system (1) in the transverse case reduces to the system

$$
u_x = \frac{1}{\varepsilon}v_y, \qquad u_y = -\frac{1}{\varepsilon}v_x
$$

where ε is a positive differentiable function of $r = \sqrt{x^2+y^2}$. This system describing ε-analytic functions is equivalent to the main Vekua equation (8) where $f = \sqrt{\varepsilon}$. In order to apply theorem 6 we denote $u = \ln r$ and $U(u) = \sqrt{\varepsilon(e^u)}$. Then taking $V \equiv 1$ we obtain the generating pair (F,G) for equation (8) in the desirable form

$$
F = U(u), \qquad G = \frac{i}{U(u)}. \tag{12}
$$

The analytic function Φ (from theorem 6) corresponding to the polar coordinate system has the form $\Phi(z) = \ln z$ and consequently $\Phi_z(z) = 1/z$. We note that Φ_z has a pole in the origin and a zero at infinity. Thus, theorem 6 is applicable in any domain Ω which does not include these two points. Moreover, as for constructing formal powers we need to use the recursive integration defined by (10) in what follows we require Ω to be any bounded simply connected domain not containing the origin.

From theorem 6 we have that a generating sequence corresponding to the generating pair (12) can be defined as follows

$$
F_m = \frac{U}{z^m} \quad \text{and} \quad G_m = \frac{i}{z^m U} \quad \text{for even } m
$$

and

$$
F_m = \frac{1}{z^m U} \quad \text{and} \quad G_m = \frac{iU}{z^m} \quad \text{for odd } m.
$$

As was explained in subsection 3.1 in order to have a complete system of formal powers for each n we need to construct $Z^{(n)}(1, z_0; z)$ and $Z^{(n)}(i, z_0; z)$.

For $n = 0$ we have

$$Z^{(0)}(1, z_0; z) = \lambda_1^{(0)} F(z) + \mu_1^{(0)} G(z)$$

and

$$Z^{(0)}(i, z_0; z) = \lambda_i^{(0)} F(z) + \mu_i^{(0)} G(z)$$

where $\lambda_1^{(0)}$, $\mu_1^{(0)}$ are real constants chosen so that

$$\lambda_1^{(0)} F(z_0) + \mu_1^{(0)} G(z_0) = 1$$

and $\lambda_i^{(0)}$, $\mu_i^{(0)}$ are real constants such that

$$\lambda_i^{(0)} F(z_0) + \mu_i^{(0)} G(z_0) = i.$$

Taking into account that F is real and G is imaginary we obtain that

$$\lambda_1^{(0)} = \frac{1}{F(z_0)}, \qquad \mu_1^{(0)} = 0,$$

$$\lambda_i^{(0)} = 0, \qquad \mu_i^{(0)} = F(z_0).$$

Thus,

$$Z^{(0)}(1, z_0; z) = \frac{F(z)}{F(z_0)} = \sqrt{\frac{\varepsilon(r)}{\varepsilon(r_0)}}$$

and

$$Z^{(0)}(i, z_0; z) = \frac{iF(z_0)}{F(z)} = i\sqrt{\frac{\varepsilon(r_0)}{\varepsilon(r)}}$$

where $r_0 = |z_0|$.

For constructing $Z^{(1)}(1, z_0; z)$ and $Z^{(1)}(i, z_0; z)$ we need first the formal powers $Z_1^{(0)}(1, z_0; z)$ and $Z_1^{(0)}(i, z_0; z)$. According to definition 3 they have the form

$$Z_1^{(0)}(1, z_0; z) = \lambda_1^{(1)} F_1(z) + \mu_1^{(1)} G_1(z)$$

and

$$Z_1^{(0)}(i, z_0; z) = \lambda_i^{(1)} F_1(z) + \mu_i^{(1)} G_1(z)$$

where $\lambda_1^{(1)}$, $\mu_1^{(1)}$ are real numbers such that

$$\lambda_1^{(1)} F_1(z_0) + \mu_1^{(1)} G_1(z_0) = 1$$

and $\lambda_i^{(1)}$, $\mu_i^{(1)}$ are real numbers such that

$$\lambda_i^{(1)} F_1(z_0) + \mu_i^{(1)} G_1(z_0) = i.$$

Thus in order to determine $\lambda_1^{(1)}$, $\mu_1^{(1)}$ and $\lambda_i^{(1)}$, $\mu_i^{(1)}$ we should solve two systems of linear algebraic equations:

$$\lambda_1^{(1)} \frac{1}{z_0 \varepsilon^{1/2}(r_0)} + \mu_1^{(1)} \frac{i\varepsilon^{1/2}(r_0)}{z_0} = 1$$

and

$$\lambda_i^{(1)} \frac{1}{z_0 \varepsilon^{1/2}(r_0)} + \mu_i^{(1)} \frac{i\varepsilon^{1/2}(r_0)}{z_0} = i$$

which can be rewritten as follows

$$\lambda_1^{(1)} + \mu_1^{(1)} i\varepsilon(r_0) = \varepsilon^{1/2}(r_0) z_0$$

and

$$\lambda_i^{(1)} + \mu_i^{(1)} i\varepsilon(r_0) = i\varepsilon^{1/2}(r_0) z_0.$$

From here we obtain

$$\lambda_1^{(1)} = \varepsilon^{1/2}(r_0) x_0, \quad \mu_1^{(1)} = \varepsilon^{-1/2}(r_0) y_0, \quad \lambda_i^{(1)} = -\varepsilon^{1/2}(r_0) y_0, \quad \mu_i^{(1)} = \varepsilon^{-1/2}(r_0) x_0.$$

Let us notice that in general for odd m we have

$$Z_m^{(0)}(1, z_0; z) = \frac{\lambda_1^{(m)}}{z^m \varepsilon^{1/2}(r)} + \frac{i\mu_1^{(m)} \varepsilon^{1/2}(r)}{z^m},$$

$$Z_m^{(0)}(i, z_0; z) = \frac{\lambda_i^{(m)}}{z^m \varepsilon^{1/2}(r)} + \frac{i\mu_i^{(m)} \varepsilon^{1/2}(r)}{z^m}$$

where

$$\lambda_1^{(m)} = \varepsilon^{1/2}(r_0) \operatorname{Re} z_0^m = \varepsilon^{1/2}(r_0) r_0^m \cos m\theta_0,$$
$$\mu_1^{(m)} = \varepsilon^{-1/2}(r_0) \operatorname{Im} z_0^m = \varepsilon^{-1/2}(r_0) r_0^m \sin m\theta_0,$$
$$\lambda_i^{(m)} = -\varepsilon^{1/2}(r_0) \operatorname{Im} z_0^m = -\varepsilon^{1/2}(r_0) r_0^m \sin m\theta_0,$$

$$\mu_i^{(m)} = \varepsilon^{-1/2}(r_0)\operatorname{Re} z_0^m = \varepsilon^{-1/2}(r_0) r_0^m \cos m\theta_0,$$

θ_0 is the argument of the complex number z_0.

Thus, for odd m:

$$Z_m^{(0)}(1, z_0; z) = \left(\frac{r_0}{z}\right)^m \left(\cos m\theta_0 \sqrt{\frac{\varepsilon(r_0)}{\varepsilon(r)}} + i \sin m\theta_0 \sqrt{\frac{\varepsilon(r)}{\varepsilon(r_0)}}\right),$$

$$Z_m^{(0)}(i, z_0; z) = \left(\frac{r_0}{z}\right)^m \left(-\sin m\theta_0 \sqrt{\frac{\varepsilon(r_0)}{\varepsilon(r)}} + i \cos m\theta_0 \sqrt{\frac{\varepsilon(r)}{\varepsilon(r_0)}}\right).$$

In a similar way we obtain the corresponding formulas for even m:

$$Z_m^{(0)}(1, z_0; z) = \left(\frac{r_0}{z}\right)^m \left(\cos m\theta_0 \sqrt{\frac{\varepsilon(r)}{\varepsilon(r_0)}} + i \sin m\theta_0 \sqrt{\frac{\varepsilon(r_0)}{\varepsilon(r)}}\right),$$

$$Z_m^{(0)}(i, z_0; z) = \left(\frac{r_0}{z}\right)^m \left(-\sin m\theta_0 \sqrt{\frac{\varepsilon(r)}{\varepsilon(r_0)}} + i \cos m\theta_0 \sqrt{\frac{\varepsilon(r_0)}{\varepsilon(r)}}\right).$$

In order to apply formula (10) for constructing formal powers of higher orders we need to calculate the adjoint generating pairs (F_m^*, G_m^*). For odd m we have

$$F_m^* = -\frac{iz^m}{\varepsilon^{1/2}(r)}, \quad G_m^* = \varepsilon^{1/2}(r) z^m.$$

For even m we obtain

$$F_m^* = -iz^m \varepsilon^{1/2}(r), \quad G_m^* = \frac{z^m}{\varepsilon^{1/2}(r)}.$$

Now the whole procedure of construction of formal powers can be easily algorithmized. The obtained system of formal powers

$$\left\{Z^{(n)}(1, z_0; z), \quad Z^{(n)}(i, z_0; z)\right\}_{n=0}^{\infty}$$

is complete in the space of all solutions of the main Vekua equation (8) with $f = \varepsilon^{1/2}(r)$, i.e., any regular solution W of (8) in Ω can be represented in the form of a normally convergent series

$$W(z) = \sum_{n=0}^{\infty} Z^{(n)}(a_n, z_0; z) = \sum_{n=0}^{\infty} \left(a_n' Z^{(n)}(1, z_0; z) + a_n'' Z^{(n)}(i, z_0; z)\right)$$

where $a_n' = \operatorname{Re} a_n$, $a_n'' = \operatorname{Im} a_n$ and z_0 is an arbitrary fixed point in Ω.

References

[1] Aleksandrov A Ya and Solovyev Yu I 1978 Spatial problems of elasticity theory: application of methods of the theory of functions of complex variable Moscow: Nauka (in Russian).

[2] Bers L 1952 Theory of pseudo-analytic functions. New York University.

[3] Bers L 1956 Formal powers and power series. Communications on Pure and Applied Mathematics **9** 693-711.

[4] Chemeris V S 1995 Construction of Cauchy integrals for one class of nonanalytic functions of complex variables. J. Math. Sci. **75**, no. 4, 1857–1865.

[5] Courant R and Hilbert D 1989 Methods of Mathematical Physics, v. 2. Wiley-Interscience.

[6] Demidenko Eu. 2006 Separable Laplace equation, magic Toeplitz matrix, and generalized Ohm's law. Applied Mathematics and Computation **181**, 1313-1327.

[7] Fryant A 1981 Ultraspherical expansions and pseudo analytic functions. Pacific Journal of Math. **94**, No. 1, 83-105.

[8] Goman O G 1984 Representation in terms of p-analytic functions of the general solution of equations of the theory of elasticity of a transversely isotropic body. J. Appl. Math. Mech. **48**, 62-67.

[9] Ismailov A Ja and Tagieva M A 1970 On the representation of generalized analytic functions by a series of pseudopolynomials. Soviet Math. Dokl. **11**, No. 6, 1605-1608.

[10] Kapshivyi O O and Klen I V 1995 Solving boundary-value problems of x^k-analytic functions for a semicircle. J. Math. Sci. **75**, No. 4, 1785–1791.

[11] Kapshivyi A A and Yazkulyev M 1993 Solution of boundary-value problems of p-analytical functions with the characteristic $p = x/(x^2 + y^2)$ on a halfplane with cuts. J. Sov. Math. **66**, No.4, 2369-2376.

[12] Kravchenko V V 2005 On a relation of pseudoanalytic function theory to the two-dimensional stationary Schrödinger equation and Taylor series in formal powers for its solutions. J. of Phys. A , **38**, No. 18, 3947-3964.

[13] Kravchenko V V 2005 On the relationship between p-analytic functions and the Schrödinger equation. Zeitschrift für Analysis und ihre Anwendungen, **24**, No. 3, 487-496.

[14] Kravchenko V V 2006 On a factorization of second order elliptic operators and applications. Journal of Physics A: Mathematical and General, 2006, **39**, No. 40, 12407-12425.

[15] Kravchenko V V Recent developments in applied pseudoanalytic function theory. To appear, available from www.arxiv.org.

[16] Menke K 1974 Zur Approximation pseudoanalytischer Funktionen durch Pseudopolynome. Manuscripta Math. **11** 111-125.

[17] Polozhy G N 1965 Generalization of the theory of analytic functions of complex variables: p-analytic and (p, q)-analytic functions and some applications. Kiev University Publishers (in Russian).

[18] Zabarankin M and Ulitko A F 2006 Hilbert formulas for r-analytic functions in the domain exterior to spindle. SIAM Journal of Applied Mathematics **66**, No. 4, 1270-1300.